二変数関数の Taylor 展開

$\alpha.\ \beta$ が小さいとき，

$$y \sim y_0 + \left(\frac{\partial y}{\partial \alpha}\right)_0 \alpha + \left(\frac{\partial y}{\partial \beta}\right)_0 \beta + \frac{1}{2}\left(\frac{\partial^2 y}{\partial \alpha^2}\right)_0 \alpha^2 + \left(\frac{\partial^2 y}{\partial \alpha \partial \beta}\right)_0 \alpha\beta + \frac{1}{2}\left(\frac{\partial^2 y}{\partial \beta^2}\right)_0 \beta^2$$

と近似できます。ここで $\alpha = \beta = 0$ のときの y，$\frac{\partial y}{\partial \alpha}$，$\frac{\partial y}{\partial \beta}$，$\frac{\partial^2 y}{\partial \alpha^2}$，$\frac{\partial^2 y}{\partial \alpha \partial \beta}$，$\frac{\partial^2 y}{\partial \beta^2}$ の値をそれぞれ y_0，$\left(\frac{\partial y}{\partial \alpha}\right)_0$，$\left(\frac{\partial y}{\partial \beta}\right)_0$，$\left(\frac{\partial^2 y}{\partial \alpha^2}\right)_0$，$\left(\frac{\partial^2 y}{\partial \alpha \partial \beta}\right)_0$，$\left(\frac{\partial^2 y}{\partial \beta^2}\right)_0$ とします。　このことは，以下のように導くことができます。まず，

$$\begin{aligned} y &= (y)_{\alpha=0} + \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} \alpha + \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=0} \frac{\alpha^2}{2} + \left(\frac{\partial^3 y}{\partial \alpha^3}\right)_{\alpha=0} \frac{\alpha^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\partial^{(n)} y}{\partial \alpha^{(n)}}\right)_{\alpha=0} \frac{\alpha^n}{n!} \end{aligned}$$

と書けますが，

$$\begin{aligned} (y)_{\alpha=0} &= (y)_{\alpha=\beta=0} + \left(\frac{\partial y}{\partial \beta}\right)_{\alpha=\beta=0} \beta + \left(\frac{\partial^2 y}{\partial \beta^2}\right)_{\alpha=\beta=0} \frac{\beta^2}{2} + \left(\frac{\partial^3 y}{\partial \beta^3}\right)_{\alpha=\beta=0} \frac{\beta^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\partial^{(n)} y}{\partial \beta^{(n)}}\right)_{\alpha=\beta=0} \frac{\beta^n}{n!} \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} &= \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=\beta=0} + \left(\frac{\partial^2 y}{\partial \alpha \partial \beta}\right)_{\alpha=\beta=0} \beta + \left(\frac{\partial^3 y}{\partial \alpha \partial \beta^2}\right)_{\alpha=\beta=0} \frac{\beta^2}{2} + \left(\frac{\partial^4 y}{\partial \alpha \partial \beta^3}\right)_{\alpha=\beta=0} \frac{\beta^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\partial^{(n+1)} y}{\partial \alpha \partial \beta^{(n)}}\right)_{\alpha=\beta=0} \frac{\beta^n}{n!} \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=0} &= \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=\beta=0} + \left(\frac{\partial^3 y}{\partial \alpha^2 \partial \beta}\right)_{\alpha=\beta=0} \beta + \left(\frac{\partial^4 y}{\partial \alpha^2 \partial \beta^2}\right)_{\alpha=\beta=0} \frac{\beta^2}{2} + \left(\frac{\partial^5 y}{\partial \alpha^2 \partial \beta^3}\right)_{\alpha=\beta=0} \frac{\beta^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\partial^{(n+2)} y}{\partial \alpha^2 \partial \beta^{(n)}}\right)_{\alpha=\beta=0} \frac{\beta^n}{n!} \end{aligned}$$

などの関係から，

$$\begin{aligned} y &= (y)_{\alpha=0} + \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=0} \alpha + \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=0} \frac{\alpha^2}{2} + \left(\frac{\partial^3 y}{\partial \alpha^3}\right)_{\alpha=0} \frac{\alpha^3}{3!} + \cdots \\ &= (y)_{\alpha=\beta=0} + \left(\frac{\partial y}{\partial \beta}\right)_{\alpha=\beta=0} \beta + \left(\frac{\partial^2 y}{\partial \beta^2}\right)_{\alpha=\beta=0} \frac{\beta^2}{2} + \left(\frac{\partial^3 y}{\partial \beta^3}\right)_{\alpha=\beta=0} \frac{\beta^3}{3!} + \cdots \\ &\qquad + \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=\beta=0} \alpha + \left(\frac{\partial^2 y}{\partial \alpha \partial \beta}\right)_{\alpha=\beta=0} \alpha\beta + \left(\frac{\partial^3 y}{\partial \alpha \partial \beta^2}\right)_{\alpha=\beta=0} \frac{\alpha\beta^2}{2} + \cdots \\ &\qquad + \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=\beta=0} \frac{\alpha^2}{2} + \left(\frac{\partial^3 y}{\partial \alpha^2 \partial \beta}\right)_{\alpha=\beta=0} \frac{\alpha^2\beta}{2} + \left(\frac{\partial^4 y}{\partial \alpha^2 \partial \beta^2}\right)_{\alpha=\beta=0} \frac{\alpha^2\beta^2}{2 \cdot 2} + \cdots \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\partial^{(m+n)} y}{\partial \alpha^{(m)} \partial^{(n)} \beta}\right)_{\alpha=\beta=0} \frac{\alpha^m \beta^n}{m!n!} \end{aligned}$$

となります。これを並べ替えれば

$$
\begin{aligned}
y = (y)_{\alpha=\beta=0} &+ \left(\frac{\partial y}{\partial \alpha}\right)_{\alpha=\beta=0} \alpha + \left(\frac{\partial y}{\partial \beta}\right)_{\alpha=\beta=0} \beta \\
&+ \left(\frac{\partial^2 y}{\partial \alpha^2}\right)_{\alpha=\beta=0} \frac{\alpha^2}{2} + \left(\frac{\partial^2 y}{\partial \alpha \partial \beta}\right)_{\alpha=\beta=0} \alpha\beta + \left(\frac{\partial^2 y}{\partial \beta^2}\right)_{\alpha=\beta=0} \frac{\beta^2}{2} \\
&+ \left(\frac{\partial^3 y}{\partial \alpha^3}\right)_{\alpha=\beta=0} \frac{\alpha^2}{3!} + \left(\frac{\partial^3 y}{\partial \alpha^2 \partial \beta}\right)_{\alpha=\beta=0} \frac{\alpha^2\beta}{2} + \left(\frac{\partial^3 y}{\partial \alpha \partial \beta^2}\right)_{\alpha=\beta=0} \frac{\alpha\beta^2}{2} + \left(\frac{\partial^3 y}{\partial \beta^3}\right)_{\alpha=\beta=0} \frac{\beta^2}{3!} \\
&+ \cdots \\
= \sum_{l=0}^{\infty} \sum_{n=0}^{l} & \left(\frac{\partial^{(l)} y}{\partial \alpha^{(l-n)} \partial^{(n)} \beta}\right)_{\alpha=\beta=0} \frac{\alpha^{l-n}\beta^n}{(l-n)!n!}
\end{aligned}
$$

と書けるわけです。